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RESEARCH PAPER

DIFFERENTIAL AND INTEGRAL RELATIONS IN THE CLASS OF MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

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Abstract

As recently observed by Bazhlekova and Dimovski [1], the n-th derivative of the 2-parametric Mittag-Leffler function gives a 3-parametric Mittag-Leffler function, known as the Prabhakar function. Following this analogy, the n-th derivative of the ($2m-$index) multi-index Mittag-Leffler functions [6] is obtained, and it turns out that it is expressed in terms of the ($3m-$index) Mittag-Leffler functions [10, 11].

Further, some special cases of the fractional order Riemann–Liouville and Erdélyi–Kober integrals of the Mittag-Leffler functions are calculated and interesting relations are proved. Analogous relations happen to connect the $3m$-Mittag-Leffler functions with the integrals and derivatives of $2m$-Mittag-Leffler functions.

Finally, multiple Erdélyi–Kober fractional integration operators, as operators of the generalized fractional calculus [5], are shown to relate the $2m$- and $3m$-parametric Mittag-Leffler functions.

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Key Words and Phrases: fractional calculus, Mittag-Leffler functions, multi-index Mittag-Leffler functions, Riemann–Liouville and Erdélyi–Kober fractional integral and derivative

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1. Introduction

The Mittag-Leffler (ML) function (2-parametric) is defined by the power series which is an entire function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (z \in \mathbb{C};\ \alpha,\beta \in \mathbb{C},\ Re(\alpha) > 0). \tag{1.1}$$

Arising in the beginning of 20th century (initially for $\beta = 1$) it remained unremarked and unused for a long time (almost for a half of century). However, nowadays it enjoys wide range applications and many generalizations. Among them is the 3-parametric Mittag-Leffler function

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \qquad (z \in \mathbb{C};\ \alpha,\beta,\gamma \in \mathbb{C},\ Re(\alpha) > 0), \tag{1.2}$$

introduced by Prabhakar [12] and known in the literature also as the Prabhakar function.

Other generalizations are the multi-index Mittag-Leffler functions. First of them is the multi-index ML function with $2m$ parameters $(m = 1, 2, 3, ...)$:

$$E_{(\alpha_i),(\beta_i)}(z) = E^{m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)}. \tag{1.3}$$

It was introduced by Luchko and Yakubovich [14] and Kiryakova [6] and studied in details by Kiryakova [6, 7]. Originally defined for $\alpha_i > 0$ and β_i arbitrary real (complex) numbers, later its definition is extended for complex parameters with $Re(\alpha_i) > 0$ (see e.g. [6] for the first case and the works [2, 3] for the second one). For the applications of this class of special functions in the solutions of fractional order differential equations and models, see e.g. in Kiryakova and Luchko [8]. The survey by Kilbas et al. [4] describes the historical development of the theory of these multi-index ($2m$-parametric) Mittag-Leffler functions as a subclass of the Wright generalized hypergeometric functions ${}_p\Psi_q(z)$. The method of Mellin–Barnes type integral representations allows these authors to extend the considered functions and to study them in the case of arbitrary values of parameters.

The next is the multi-index ML function defined by means of $3m$ parameters:

$$E^{(\gamma_i)}_{(\alpha_i),(\beta_i)}(z) = E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k\dots(\gamma_m)_k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)} \frac{z^k}{(k!)^m}$$
$$(z \in \mathbb{C};\ \alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{C},\ Re(\alpha_i) > 0), \tag{1.4}$$

and was introduced by the author in [10, 11]. It generalizes both the 3-parametric Prabhakar function (1.2) and the $2m$-parametric ML function (1.3).

2. Integer order derivatives of ML type functions

Recently, it has been obtained by Bazhlekova and Dimovski [1] that the n-th derivative of the 2-parametric Mittag-Leffler function gives a 3-parametric Mittag-Leffler function, introduced by Prabhakar (up to a constant), namely

$$E^{(n)}_{\alpha,\beta}(z) = n!\, E^{n+1}_{\alpha,\beta+n\alpha}(z). \tag{2.1}$$

Following this analogy, the n-th derivative of the ($2m$-parametric) multi-index Mittag-Leffler function (1.3) is calculated. It turns out that this integer order derivative is expressed by the $3m$-parametric Mittag-Leffler function (1.4), also up to a constant.

As usual, by

$$D^n = \frac{d^n}{dz^n} = \left(\frac{d}{dz}\right)^n, \quad n \in \mathbb{N}_0,$$

we denote the n-tuple differentiation with the conventional assumption that $D^n f(z) = f(z)$ when $n = 0$, i.e. $D^0 f(z) = f(z)$. The next assertion gives the relation between the functions (1.4) and the n-th derivative of (1.3).

Theorem 2.1. *Let $\alpha_i, \beta_i, z \in \mathbb{C}$ and let $Re(\alpha_i) > 0$ for $i = 1, \dots, m$. Then the following equality holds true for all the values of $n \in \mathbb{N}_0$:*

$$D^n \left[E_{(\alpha_i),(\beta_i)}(z)\right] = \frac{d^n}{dz^n}\left[E_{(\alpha_i),(\beta_i)}(z)\right] = \Gamma(n+1)\, E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z), \tag{2.2}$$

with $\gamma_1 = n+1$, $\gamma_2 = \dots = \gamma_n = 1$.

P r o o f. Having in mind that

$$D^n\left(z^k\right) = \frac{\Gamma(k+1)}{\Gamma(k-n+1)}\, z^{k-n},$$

one obtains

$$D^n \left[E_{(\alpha_i),(\beta_i)}(z)\right] = \sum_{k=n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \frac{z^{k-n}}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{\Gamma(k+1)} \frac{z^k}{\Gamma(\alpha_1 k+\alpha_1 n+\beta_1)\dots\Gamma(\alpha_m k+\alpha_m n+\beta_m)}.$$

Thus, taking a term-by-term differentiation under the summation sign (which is possible in accordance with the uniform convergence of the series (1.3) in any compact subset of $\mathbb{C}$ and the differentiability of the power function), the proof ends, by using that

$$\Gamma(k+n+1) = (n+1)_k\, \Gamma(n+1) \ \text{ and } \ \frac{\Gamma(k+n+1)}{\Gamma(k+1)} = \Gamma(n+1)\, \frac{(n+1)_k\,(1)_k \dots (1)_k}{(k!)^m}.$$

If $n = 0$, we have

$$D^0\left[E_{(\alpha_i),(\beta_i)}(z)\right]=E_{(\alpha_i),(\beta_i)}(z)=E^{(1),m}_{(\alpha_i),(\beta_i)}(z)=\Gamma(n+1)\,E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z),$$

with $\gamma_1 = n+1 = 1$, $\gamma_2 = \cdots = \gamma_n = 1$, therefore (2.2) is also fulfilled. □

3. Fractional Riemann–Liouville integrals and derivatives

The most popular definition for integration of arbitrary (i.e. not obligatorily integer) order $\lambda \in \mathbb{C}$ ($Re(\lambda) > 0$) is the *Riemann–Liouville (R-L) fractional integral* [13]

$$R^\lambda f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z (z-t)^{\lambda-1} f(t)dt = \frac{z^\lambda}{\Gamma(\lambda)}\int_0^1 (1-\tau)^{\lambda-1} f(z\tau)d\tau\,. \quad (3.1)$$

The corresponding *Riemann–Liouville fractional derivative* of order λ is defined as a composition of a derivative of integer order and an integral of fractional order of the form (3.1), namely:

$$D^\lambda f(z) := D^n\, R^{n-\lambda} f(z), \quad (3.2)$$

where $n := [Re(\lambda)] + 1 > Re(\lambda)$, $[Re(\lambda)]$ = integer part of $Re(\lambda)$.

In this section we consider the Riemann–Liouville (R-L) fractional integrals and derivatives (3.1) and (3.2) of the multi-index Mittag-Leffler function (1.3), multiplied by a suitable power function. For the simplest case $m = 1$ of the classical ML function with 2 parameters (1.1) the following elementary assertion can be formulated.

Theorem 3.1. *Let α, β, $\widetilde{\lambda} \in \mathbb{C}$, $Re(\alpha) > 0$, $0 < Re(\widetilde{\lambda}) < 1$, $n \in \mathbb{N}$, and let $\lambda = n - 1 + \widetilde{\lambda}$. Then*

$$D^\lambda[z^{\widetilde{\lambda}} E_{\alpha,\beta}(z)] = \Gamma(\lambda+1) E^{\lambda+1}_{\alpha,\beta+(n-1)\alpha}(z) \quad (|\arg z| < \pi). \quad (3.3)$$

P r o o f. We put $p = k + \widetilde{\lambda}$ in the well-known formula [13]

$$D^\lambda\,(z^p) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\, z^{p-\lambda}, \quad (3.4)$$

and exchanging again the orders of differentiation and summation (justified by the differentiability of the power function for $|\arg z| < \pi$ and uniform convergence of the series in any compact subset of this set), we obtain

$$D^\lambda\left[z^{\widetilde{\lambda}} E_{\alpha,\beta}(z)\right] = D^\lambda\left[\sum_{k=0}^{\infty}\frac{z^{k+\widetilde{\lambda}}}{\Gamma(\alpha k+\beta)}\right] = \cdots$$

$$= \sum_{k=0}^{\infty}\frac{(\widetilde{\lambda}+n)_k\,\Gamma(\widetilde{\lambda}+n)}{\Gamma(\alpha k+(n-1)\alpha+\beta)}\,\frac{z^k}{k!} = \sum_{k=0}^{\infty}\frac{(\lambda+1)_k\,\Gamma(\lambda+1)}{\Gamma(\alpha k+(n-1)\alpha+\beta)}\,\frac{z^k}{k!}\,,$$

since $\widetilde{\lambda} + n = \widetilde{\lambda} + n - 1 + 1 = \lambda + 1$. Thus we get (3.3). □

In particular, if we set $n = 1$ in Theorem 3.1 then the parameter λ becomes $\lambda = \widetilde{\lambda}$ and (3.3) gives the following corollary.

Corollary 3.1. *If α, β, $\lambda \in \mathbb{C}$, $Re(\alpha) > 0$ and $0 < Re(\lambda) < 1$, then the following identity holds*

$$D^{\lambda}[z^{\lambda} E_{\alpha,\beta}(z)] = \Gamma(\lambda+1) E^{\lambda+1}_{\alpha,\beta}(z) \quad (|\arg z| < \pi). \tag{3.5}$$

For the multi-index Mittag-Leffler functions, the result is given as below.

Theorem 3.2. *Let α_i, β_i, $\widetilde{\lambda} \in \mathbb{C}$ and $Re(\alpha_i) > 0$ for $i = 1, \dots, m$, and let also $0 < Re(\widetilde{\lambda}) < 1$, $n \in \mathbb{N}$, $\lambda = n - 1 + \widetilde{\lambda}$. Then it holds*

$$D^{\lambda}[z^{\widetilde{\lambda}} E_{(\alpha_i),\,(\beta_i)}(z)] = \Gamma(\lambda+1) E^{(\gamma_i)}_{(\alpha_i),\,(\beta_i+(n-1)\alpha_i)}(z) \quad (|\arg z| < \pi), \tag{3.6}$$

with parameters $\gamma_1 = \lambda + 1, \gamma_2 = \dots = \gamma_m = 1$.

P r o o f. From the definition (1.3) we have

$$z^{\widetilde{\lambda}} E_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^{k+\widetilde{\lambda}}}{\Gamma(\alpha_1 k + \beta_1) \dots \Gamma(\alpha_m k + \beta_m)},$$

and then, in accordance with (3.4) and by analogy with the proof of Theorem 3.1, it consequently follows that

$$D^{\lambda}[z^{\widetilde{\lambda}} E_{(\alpha_i),\,(\beta_i)}(z)] = \sum_{k=n-1}^{\infty} \frac{\Gamma(k+\widetilde{\lambda}+1)}{\Gamma(k-n+2)} \, \frac{z^{k+1-n}}{\Gamma(\alpha_1 k + \beta_1) \dots \Gamma(\alpha_m k + \beta_m)}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+\widetilde{\lambda}+n)}{\Gamma(k+1)} \, \frac{z^k}{\Gamma(\alpha_1 k + \widetilde{\beta}_1) \dots \Gamma(\alpha_m k + \widetilde{\beta}_m)}$$

$$= \Gamma(\lambda+1) \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{\Gamma(\alpha_1 k + \widetilde{\beta}_1) \dots \Gamma(\alpha_m k + \widetilde{\beta}_m)} \, \frac{z^k}{k!},$$

where $\widetilde{\beta}_i = (n-1)\alpha_i + \beta_i$ for $i = 1, \dots, m$. □

Corollary 3.2. *If α_i, β_i, $\lambda \in \mathbb{C}$, $Re(\alpha_i) > 0$ for $i = 1, \dots, m$, $0 < Re(\lambda) < 1$, then the following equality holds:*

$$D^{\lambda}[z^{\lambda} E_{(\alpha_i),\,(\beta_i)}(z)] = \Gamma(\lambda+1) E^{(\gamma_i)}_{(\alpha_i),\,(\beta_i)}(z) \quad (|\arg z| < \pi), \tag{3.7}$$

with parameters $\gamma_1 = \lambda + 1, \gamma_2 = \dots = \gamma_m = 1$.

P r o o f. Analogously to Corollary 3.1, the truth of (3.7) automatically follows from (3.6), by taking $n = 1$. □

THEOREM 3.3. *Let α, β, $\lambda \in \mathbb{C}$, and let $Re(\alpha) > 0$ and $0 < Re(\lambda) < 1$. Then*

$$R^\lambda[z^{-\lambda} E_{\alpha,\,\beta}(\omega z)] = \Gamma(1-\lambda)\, E^{1-\lambda}_{\alpha,\,\beta}(\omega z) \quad (|\arg z| < \pi). \tag{3.8}$$

P r o o f. Putting $f(z) = z^{-\lambda} E_{\alpha,\,\beta}(\omega z)$, we have

$$f(z\tau) = z^{-\lambda}\tau^{-\lambda} E_{\alpha,\,\beta}(\omega z\tau) = z^{-\lambda}\tau^{-\lambda} \sum_{k=0}^{\infty} \frac{(\omega z\tau)^k}{\Gamma(\alpha k + \beta)},$$

whence, using the second part of definition (3.1), we obtain

$$\begin{aligned} R^\lambda f(z) &= \frac{1}{\Gamma(\lambda)} \int_0^1 (1-\tau)^{\lambda-1}\tau^{-\lambda} \sum_{k=0}^{\infty} \frac{(\omega z\tau)^k}{\Gamma(\alpha k+\beta)}\, d\tau \\ &= \sum_{k=0}^{\infty} \frac{(\omega z)^k}{\Gamma(\alpha k+\beta)} \int_0^1 \frac{(1-\tau)^{\lambda-1}\tau^{k-\lambda}}{\Gamma(\lambda)}\, d\tau\,. \end{aligned}$$

Changing the orders of integration and summation is possible due to the integrability of the power function for $|\arg z| < \pi$ and uniform convergence of the series above in any compact subset of this set. Now, taking into account the relation

$$\int_0^1 \frac{(1-\tau)^{\lambda-1}\tau^{k-\lambda}}{\Gamma(\lambda)} d\tau = \frac{B(\lambda, k-\lambda+1)}{\Gamma(\lambda)} = \frac{\Gamma(k-\lambda+1)}{\Gamma(k+1)} = \frac{(1-\lambda)_k \Gamma(1-\lambda)}{k!},$$

we complete the proof of the theorem. □

THEOREM 3.4. *Let α_i, β_i, $\lambda \in \mathbb{C}$ and $Re(\alpha_i) > 0$ for $i = 1, \dots, m$. Moreover, let $0 < Re(\lambda) < 1$. Then*

$$R^\lambda[z^{-\lambda} E_{(\alpha_i),\,(\beta_i)}(\omega z)] = \Gamma(1-\lambda)\, E^{(\gamma_i)}_{(\alpha_i),\,(\beta_i)}(\omega z) \quad (|\arg z| < \pi), \tag{3.9}$$

with parameters $\gamma_1 = 1-\lambda, \gamma_2 = \dots = \gamma_m = 1$.

P r o o f. Like in the previous proof, we obtain

$$R^\lambda f(z) = \frac{1}{\Gamma(\lambda)} \int_0^1 (1-\tau)^{\lambda-1}\tau^{-\lambda} \sum_{k=0}^{\infty} \frac{(\omega z\tau)^k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)}\, d\tau$$

$$= \sum_{k=0}^{\infty} \frac{(\omega z)^k}{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)} \int_0^1 \frac{(1-\tau)^{\lambda-1} \tau^{k-\lambda}}{\Gamma(\lambda)} \, d\tau \, .$$

Then, the proof continues in the same way as this one of Theorem 3.3. □

4. Generalized Erdélyi–Kober fractional integrals and derivatives

Along with the classical Riemann–Liouville definitions of fractional order operators, the so-called Erdélyi–Kober operators are very useful in classical fractional calculus, as depending on one more parameter. Namely, for real parameters $\lambda > 0$, $\mu > 0$ and β, the *Erdélyi–Kober fractional integral operator* or order λ is defined as (for details, see e.g. Kiryakova [5]).

$$I_\mu^{\beta,\lambda} f(z) = \left[z^{-(\beta+\lambda)} R^\lambda z^\beta f(z^{1/\mu}) \right]_{z \to z^\mu}$$

$$= \frac{z^{-\mu(\beta+\lambda)}}{\Gamma(\lambda)} \int_0^z (z^\mu - t^\mu)^{\lambda-1} t^{\beta\mu} f(t) d(t^\mu) = \frac{1}{\Gamma(\lambda)} \int_0^1 (1-\tau)^{\lambda-1} \tau^\beta f(z\tau^{1/\mu}) d\tau, \tag{4.1}$$

and respectively, the *Erdélyi–Kober fractional differential operator* can be written symbolically ([5]) as

$$D_\mu^{\beta,\lambda} f(z) = \left[z^{-\beta} D^\lambda z^{\beta+\lambda} f(z^{1/\mu}) \right]_{z \to z^\mu} . \tag{4.2}$$

Note that $D_\mu^{\beta,0} f(z) = I_\mu^{\beta,0} f(z) = f(z)$, and $D_\mu^{\beta,\lambda} I_\mu^{\beta,\lambda} f(z) = f(z)$. The case with $\mu = 2$ was introduced by Sneddon, while Kober and Erdélyi originally considered the case $\mu = 1$. The Erdélyi–Kober operators (4.1) and (4.2) have numerous applications emphasized ever since Sneddon by himself, and they are widely used in FC and fractional order differential equations to model various type of physical, economical etc. processes. See for example Pagnini [9], considering the so-called Erdélyi–Kober fractional diffusion and a family of diffusive processes governed by the ggBm (generalized grey Brownian motion).

Further, we consider the operators (4.1) and (4.2) with $\mu = 1$. Then, they can be written in the forms

$$I_1^{\beta,\lambda} f(z) = z^{-(\beta+\lambda)} R^\lambda \left[z^\beta f(z) \right], \quad D_1^{\beta,\lambda} f(z) = z^{-\beta} D^\lambda \left[z^{\beta+\lambda} f(z) \right], \tag{4.3}$$

and so, these are practically Riemann–Liouville integrals and derivatives, up to power functions multipliers.

For the functions (1.1) and (1.3), first we evaluate the Erdélyi–Kober fractional integrals with parameters

$$\mu = 1, \ \lambda > 0, \ \beta = -\lambda,$$

and Erdélyi–Kober fractional derivatives with parameters

$$\mu = 1,\ \lambda = n-1+\widetilde{\lambda},\ \beta = -\lambda \quad (0<\widetilde{\lambda}<1,\ n=1,2,\dots).$$

Using the results in Section 3, we can formulate two elementary assertions.

Theorem 4.1. *Let $\alpha,\ \alpha_i,\ z \in \mathbb{C},\ i=1,2,\dots,m,\ \lambda = n-1+\widetilde{\lambda}$ $(0<\widetilde{\lambda}<1,\ m,\ n\in\mathbb{N})$, and let $|\arg z|<\pi$, $Re(\alpha)>0$, $Re(\alpha_i)>0$. Then*

$$D_1^{1-n,\,\lambda} E_{\alpha,\beta}(z) = \Gamma(1+\lambda)\, z^{n-1}\, E^{1+\lambda}_{\alpha,\,\beta+(n-1)\alpha}(z), \tag{4.4}$$

$$D_1^{1-n,\,\lambda} E_{(\alpha_i),(\beta_i)}\,(z) = \Gamma(1+\lambda)\, z^{n-1}\, E^{(\gamma_i),m}_{(\alpha_i),\,(\beta_i+(n-1)\alpha_i)}(z) \tag{4.5}$$

with parameters

$$\gamma_1 = 1+\lambda; \quad \gamma_2 = \cdots = \gamma_m = 1.$$

P r o o f. These are simple corollaries of Theorems 3.1 and 3.2. Indeed, in view of (4.3) and Theorem 3.1,

$$D_1^{1-n,\,\lambda} E_{\alpha,\beta}(z) = z^{n-1}\, D^{\lambda}\left[z^{\widetilde{\lambda}} E_{\alpha,\beta}\right] = \Gamma(1+\lambda)\, z^{n-1}\, E^{1+\lambda}_{\alpha,\,\beta+(n-1)\alpha}(z), \tag{4.6}$$

which is (4.4). The proof of (4.5) goes in a similar way. □

In particular, if $n=1$, the second representation in (4.3) and Theorem 4.1 automatically yield the result formulated below.

Corollary 4.1. *Let $\alpha,\ \alpha_i,\ z\in\mathbb{C},\ i=1,2,\dots,m,\ m\in\mathbb{N}$, and let $|\arg z|<\pi$, $Re(\alpha)>0$, $Re(\alpha_i)>0$, $0<\lambda<1$. Then*

$$D_1^{0,\,\lambda} E_{\alpha,\beta}(z) = D^{\lambda}\left[z^{\lambda} E_{\alpha,\beta}(z)\right] = \Gamma(1+\lambda)\, E^{1+\lambda}_{\alpha,\,\beta}(z), \tag{4.7}$$

$$D_1^{0,\,\lambda} E_{(\alpha_i),(\beta_i)}\,(z) = D^{\lambda}\left[z^{\lambda} E_{(\alpha_i),\,(\beta_i)}(z)\right] = \Gamma(1+\lambda)\, E^{(\gamma_i),m}_{(\alpha_i),\,(\beta_i)}(z) \tag{4.8}$$

with parameters

$$\gamma_1 = 1+\lambda; \quad \gamma_2 = \cdots = \gamma_m = 1.$$

The following assertion, which is a simple corollary of Theorems 3.3 and 3.4, refers to the operator (4.1) and it is essentially used in this section.

Theorem 4.2. *Let $\alpha,\ \alpha_i,\ \omega,\ z\in\mathbb{C},\ \omega\neq 0,\ i=1,2,\dots,m,\ m\in\mathbb{N}$, and let $|\arg z|<\pi$, $Re(\alpha)>0$, $Re(\alpha_i)>0$, $0<\lambda<1$. Then*

$$I_1^{-\lambda,\,\lambda} E_{\alpha,\beta}(\omega z) = \Gamma(1-\lambda) E^{1-\lambda}_{\alpha,\,\beta}(\omega z), \tag{4.9}$$

$$I_1^{-\lambda,\,\lambda} E_{(\alpha_i),(\beta_i)}\,(\omega z) = \Gamma(1-\lambda)\, E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(\omega z) \tag{4.10}$$

with parameters

$$\gamma_1 = 1-\lambda; \quad \gamma_2 = \cdots = \gamma_m = 1.$$

Now, we briefly recall some definitions of the operators of the so-called *Generalized Fractional Calculus* (GFC) of Kiryakova [5, 6]. The kernel function $H^{m,0}_{m,m}$ appearing below is a suitably chosen case of the Fox H-function.

Definition 4.1. Let $m \geq 1$ be an integer, $\lambda_i \geq 0$, $\mu_i > 0$, $\beta_i \in \mathbb{R}$, $i = 1, \ldots, m$. Consider $\beta = (\beta_1, \ldots, \beta_m)$ as a multi-weight and resp. $\lambda = (\lambda_1, \ldots, \lambda_m)$ as a multi-order of fractional integration. The integral operators, defined as follows:

$$I^{(\beta_i),(\lambda_i)}_{(\mu_i),m} f(z) = \begin{cases} \int\limits_0^1 H^{m,0}_{m,m}\left[\tau \left| \begin{smallmatrix} (\beta_i+\lambda_i+1-1/\mu_i, 1/\mu_i)_1^m \\ (\beta_i+1-1/\mu_i, 1/\mu_i)_1^m \end{smallmatrix} \right.\right] f(z\tau)d\tau, & \text{if } \sum\limits_{i=1}^m \lambda_i > 0, \\ f(z), \quad \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0, \end{cases} \tag{4.11}$$

are said to be multiple (m-tuple) Erdélyi–Kober fractional integration operators. More generally, the operators of the form

$$If(z) = z^{\lambda_0} I^{(\beta_i),(\lambda_i)}_{(\mu_i),m} f(z) \quad \text{with} \quad \lambda_0 \geq 0 \tag{4.12}$$

are briefly called *generalized* (m*-tuple*) *fractional integrals.*

Remark 4.1. If all $\mu_i = \mu > 0$, for $i = 1, \ldots, m$, then (4.11) is denoted by $I^{(\beta_i),(\lambda_i)}_{\mu,m} f(z)$ instead of $I^{(\beta_i),(\lambda_i)}_{(\mu_i),m} f(z)$, and the integral operators (4.11) and (4.12) reduce to such ones with the simpler G-function of Meijer:

$$I^{(\beta_i),(\lambda_i)}_{\mu,m} f(z) = \begin{cases} \int\limits_0^1 G^{m,0}_{m,m}\left[\tau \left| \begin{smallmatrix} (\beta_i+\lambda_i)_1^m \\ (\lambda_i)_1^m \end{smallmatrix} \right.\right] f(z\tau^{\frac{1}{\mu}})d\tau, & \text{if } \sum\limits_{i=1}^m \lambda_i > 0, \\ f(z), \quad \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0, \end{cases} \tag{4.13}$$

respectively

$$If(z) = z^{\lambda_0} I^{(\beta_i),(\lambda_i)}_{\mu,m} f(z) \quad \text{with} \quad \lambda_0 \geq 0. \tag{4.14}$$

For $m = 1$ the operators (4.11) turn into the classical Erdélyi–Kober integrals (4.1), for $m = 2$ these are the hypergeometric fractional integrals, for arbitrary $m > 1$ we get the hyper-Bessel integral operators as particular cases, etc. The theory of the GFC based on operators (4.11) is developed in full details in Kiryakova's book [5]. Here we briefly recall some of the basic facts we use.

Let σ be an arbitrary real number. Denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions in a complex domain Ω, starlike with respect to the origin $z = 0$, and consider the spaces

$$\mathcal{H}_\sigma(\Omega) = \{f(z) = z^p \tilde{f}(z);\ p \geq \sigma, \tilde{f}(z) \in \mathcal{H}(\Omega)\}, \quad \mathcal{H}_0(\Omega) := \mathcal{H}(\Omega).$$

For $\beta_i > -1 - \sigma/\mu_i$ and $\lambda_i > 0$ the operators (4.11) map the space $\mathcal{H}_\sigma(\Omega)$ into itself (Kiryakova [5]).

The main property of the generalized (m-tuple) fractional integrals is that the single integrals in (4.12) and (4.13), involving H- and G-functions, can be equivalently represented by means of commutative compositions of single Erdélyi–Kober integrals of the form (4.1). Namely, for $f \in \mathcal{H}_\sigma(\Omega)$ with $\beta_i > -1 - \sigma/\mu$, $\lambda_i > 0$ and $i = 1, 2, \ldots, m$:

$$\begin{gathered} I_{\mu,m}^{(\beta_i),(\lambda_i)} f(z) = \prod_{i=1}^{m} I_\mu^{\beta_i,\lambda_i} f(z) \\ = \int_0^1 \ldots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\tau_i)^{\lambda_i - 1}\ \tau_i^{\beta_i}}{\Gamma(\lambda_i)} \right] f\left((z\,\tau_1 \ldots \tau_m)^{1/\mu} \right) d\tau_1 \ldots d\tau_m. \end{gathered} \tag{4.15}$$

If some of the λ_i are zeros, e.g. $\lambda_1 = \cdots = \lambda_k = 0$, $1 \leq k \leq m$, then the corresponding multipliers are identity operators, i.e. $I_\mu^{\beta_i,\lambda_i} = I$, and the multiplicity of (4.13) reduces from m to $m - k$ (same for the order of the kernel H-functions). Decomposition relation (4.15) is the key to various applications of (4.13) and (4.14). The generalized fractional derivatives, corresponding to (4.13), are defined in [5] by differ-integral expressions analogously to the idea for (3.2) and (4.2).

Now, by analogy with Lemma 4.2 we prove a relation involving the generalized fractional integral (4.13), in this case in the class of entire functions in $\mathbb{C}$.

THEOREM 4.3. *Let $\omega \in \mathbb{C}$, $\omega \neq 0$, $0 < \lambda < 1$, and $Re(\alpha_i) > 0$ for $1 \leq i \leq m$. Then*

$$I_{1,m}^{(-\lambda_i),(\lambda_i)} E_{(\alpha_i),(\beta_i)}(\omega z) = \Gamma(1-\lambda_1) \ldots \Gamma(1-\lambda_m)\ E_{(\alpha_i),(\beta_i)}^{(1-\lambda_i),m}(\omega z).$$

P r o o f. Taking into account the following relations, based on (4.1) and (4.10),

$$\int_0^1 \frac{(1-\tau_1)^{\lambda_1 - 1}\ \tau_1^{-\lambda_1}}{\Gamma(\lambda_1)}\ E_{(\alpha_i),(\beta_i)}(\omega z \tau_1 \ldots \tau_m) d\tau_1$$

$$= I_1^{-\lambda_1,\lambda_1} E_{(\alpha_i),(\beta_i)}(\omega z \tau_2 \ldots \tau_m) = \Gamma(1-\lambda_1)\ E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}(\omega z \tau_2 \ldots \tau_m),$$

with $\gamma_1 = 1 - \lambda_1, \gamma_2 = \cdots = \gamma_m = 1$,

$$\ldots \quad \ldots \quad \ldots,$$

and respectively, denoting $\widetilde{\gamma}_1 = 1-\lambda_1, \dots, \widetilde{\gamma}_{m-1} = 1-\lambda_{m-1}, \widetilde{\gamma}_m = 1$, we deduce that

$$\int_0^1 \frac{(1-\tau_m)^{\lambda_m-1}\,\tau_m^{-\lambda_m}}{\Gamma(\lambda_m)}\,\Gamma(1-\lambda_1)\dots\Gamma(1-\lambda_{m-1})E^{(\widetilde{\gamma}_i),m}_{(\alpha_i),(\beta_i)}(\omega z\tau_m)d\tau_m$$
$$= \Gamma(1-\lambda_1)\dots\Gamma(1-\lambda_{m-1})\; I_1^{-\lambda_m,\lambda_m}\; E^{(\widetilde{\gamma}_i),m}_{(\alpha_i),(\beta_i)}(\omega z)$$
$$= \Gamma(1-\lambda_1)\dots\Gamma(1-\lambda_m)\; E^{(1-\lambda_i),m}_{(\alpha_i),(\beta_i)}(\omega z),$$

i.e. the subsequent repeated (m times) integrations in (4.15) with $\mu = 1$ lead to the identity

$$I^{(-\lambda_i),(\lambda_i)}_{1,m} E_{(\alpha_i),(\beta_i)}(\omega z) = \prod_{i=1}^{m} I_1^{-\lambda_i,\lambda_i}\; E_{(\alpha_i),(\beta_i)}(\omega z)$$
$$= \int_0^1 \dots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-\tau_i)^{\lambda_i-1}\tau_i^{-\lambda_i}}{\Gamma(\lambda_i)}\right] E_{(\alpha_i),(\beta_i)}(\omega z\tau_1\dots\tau_m)\, d\tau_1\dots d\tau_m$$
$$= \Gamma(1-\lambda_1)\dots\Gamma(1-\lambda_m)\; E^{(1-\lambda_i),m}_{(\alpha_i),(\beta_i)}(\omega z),$$

that proves the theorem. □

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